Selfish Routing
and/or
Non-atomic Congestion Games

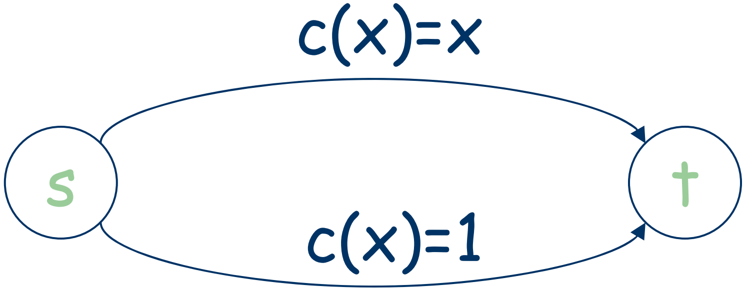
Algorithmic Game Theory Course

Co.Re.Lab. - N.T.U.A.

On this presentation we will see

- What Selfish Routing is about,
- Flows at Equilibrium and Optimal flows,
- Social welfare and the Price of Anarchy (PoA),
- Bounds on the PoA and
- How to reduce the PoA by taxing the edges

Pigou's Network



There is a source node ***s*** and a target node ***t***.

One unit of flow is to be routed from ***s*** to ***t***, using the upper and the lower edge. This unit of flow corresponds to ***infinetely many, infinitesimal players***.

- The lower edge costs constantly 1 to each player that uses her.
- The upper edge's latency for each player on her is equal to the fraction of the players using her.

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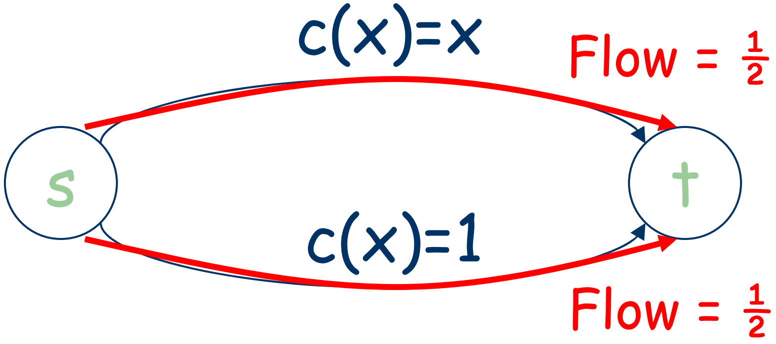

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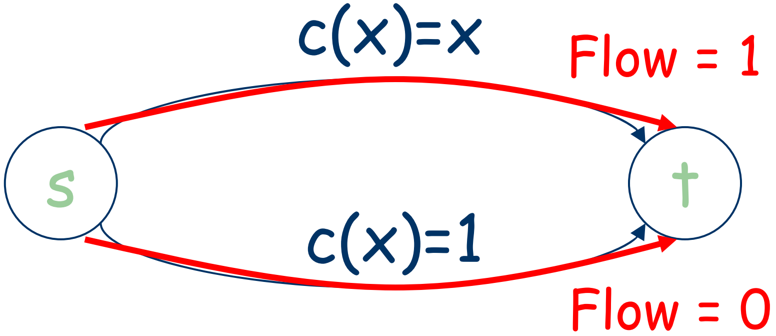

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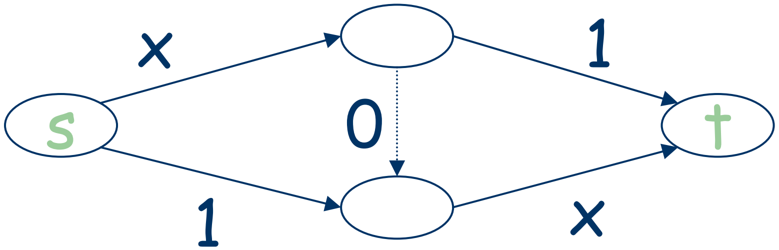
Braess' Paradox's Network



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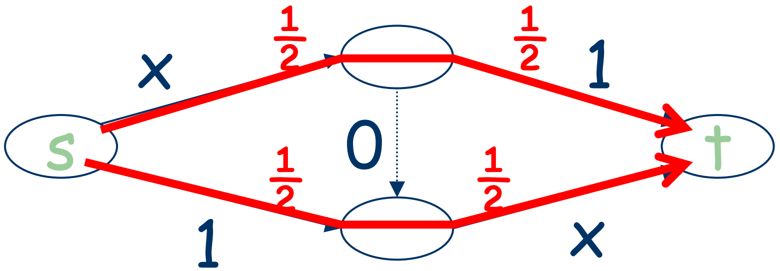

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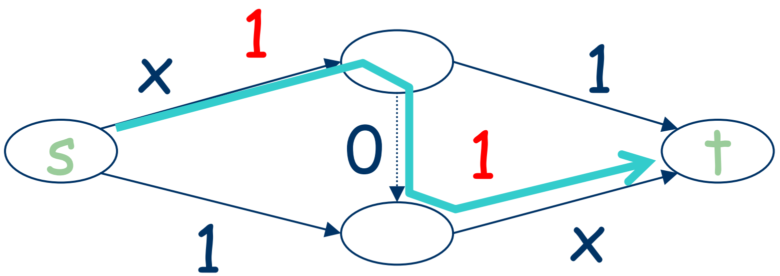


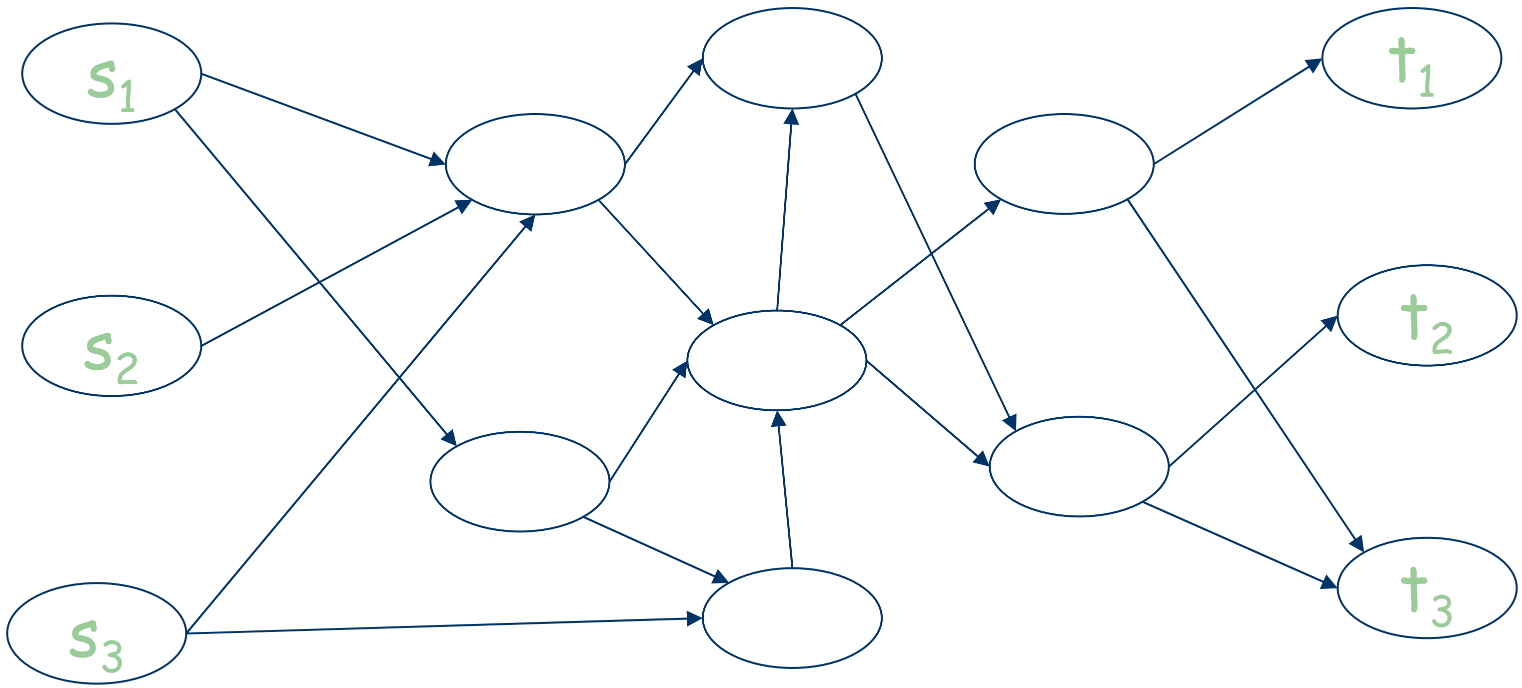
- One unit of flow is to be routed from s to t.
- The optimal routing routes half of the flow through the upper path and half of the flow through the lower path
- Players prefer the "upper-lower" path

The Mathematical Model

- a directed graph $G = (V,E)$
- k source-destination pairs $(s_1 ,t_1), \ldots, (s_k ,t_k)$
- a rate (amount) r_i of traffic from s_i to t_i
- for each edge e, a cost function $c_e(\bullet)$
  - assumed nonnegative, continuous, nondecreasing

The strategies of players with source destination pair (s_i ,t_i) are all the paths joining s_i and t_i.

Example



$r_1 = r_2 = r_3 = 1$ and for all the edges of the network $c_e(x) = x$

Flows

Let $P_i = \{p | p \text{ is a simple } s_i - t_i \text{ path}\}$ and $P = \bigcup P_i$

A flow is a function $f : P \rightarrow \Re_+$ (imagine it as a vector)

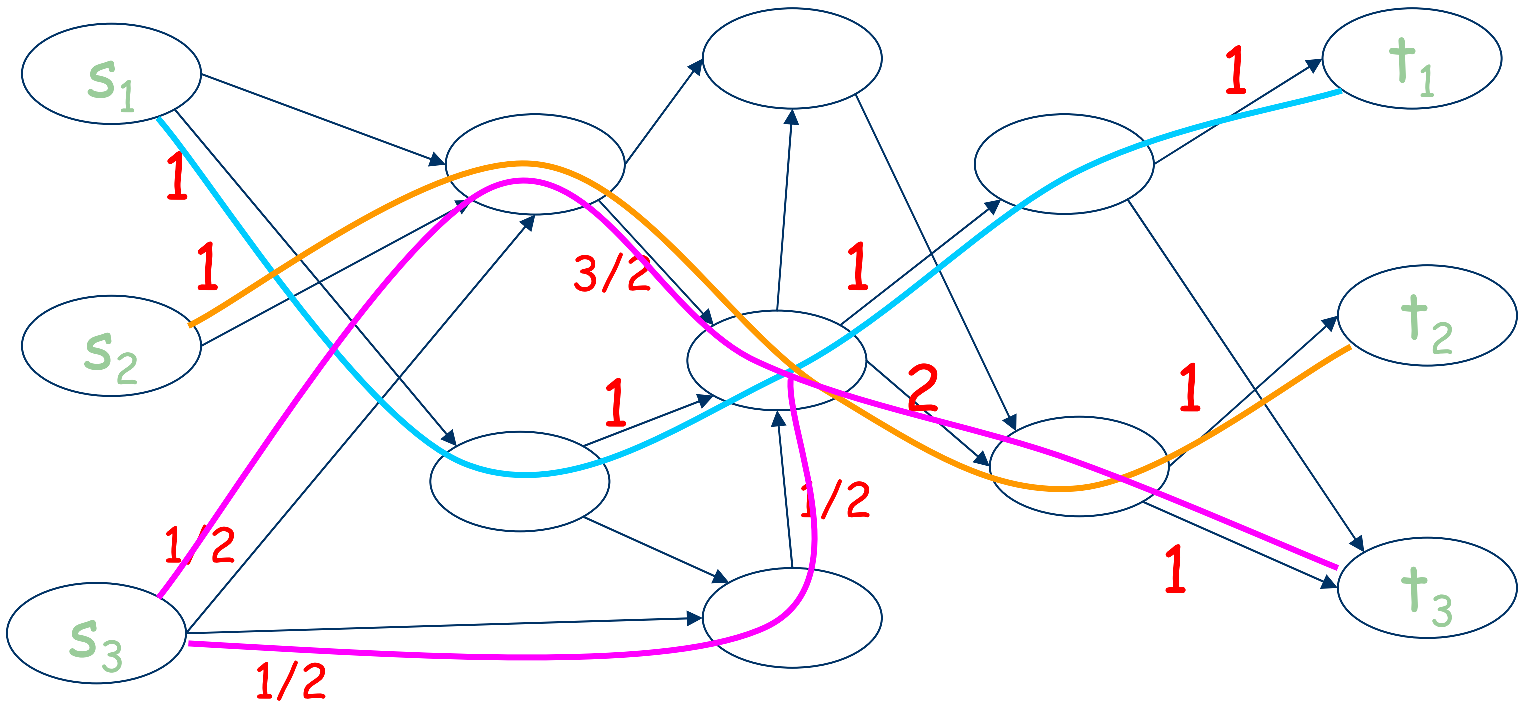
A flow is feasible if $\sum_{p \in P_i} f_p = r_i$

Edge decomposition of flow: $f_e = \sum_{p \in P : e \in p} f_p$

Each player on path *p* pays $c_p(f) = \sum_{e \in p} c_e(f_e)$

The flow's total cost is $C(f) = \sum_{p \in P} c_p(f) f_p = \sum_{e \in E} f_e c_e(f_e)$

Example



$r_1 = r_2 = r_3 = 1$ and for all the edges of the network $c_e(x) = x$

Wardrop Equilibrium (Nash flow)

A feasible flow is a Wardrop equilibrium if for every commodity *i* :

$$\forall p, q \in P_i, f_p > 0 : c_p(f) \leq c_q(f)$$

<u>Intuitively, no player has incentive to deviate</u>

Moreover: $\forall p, q \in P_i : f_p > 0, f_q > 0 \Rightarrow c_p(f) = c_q(f)$

Existence and Uniqueness

Let $\Phi(f) := \sum_{e\in E} \int_0^{f_e} c_e(x)dx$

Assume ***f*** is an equilibrium flow.

Change ***f*** to a feasible flow ***f'*** that differs with ***f*** in only two paths (p, q) of the same commodity: $f'_p = f_p - \delta,\ f'_q = f_q + \delta$

$$\Phi(f') - \Phi(f) = \sum_{e\in p\cup q} \int_0^{f'_e} c_e(x)dx - \sum_{e\in p\cup q} \int_0^{f_e} c_e(x)dx$$

$$\Downarrow$$

$$\Phi(f') - \Phi(f) = \sum_{e\in q-p} \int_{f_e}^{f_e+\delta} c_e(x)dx - \sum_{e\in p-q} \int_{f_e-\delta}^{f_e} c_e(x)dx$$

$$\Downarrow$$

$$\text{for } \delta \to 0:$$

$$\Phi(f') - \Phi(f) \approx \sum_{e\in q-p} \delta c_e(f'_e) - \sum_{e\in p-q} \delta c_e(f_e) = \delta(c_q(f') - c_p(f)) \geq 0$$

Existence and Uniqueness

Consider the convex program CP:

$$\min\ \Phi(f) := \sum_{e \in E} \int_0^{f_e} c_e(x)dx$$
so that
$$\sum_{p \in P_i} f_p = r_i, \forall i \in \{1 \ldots k\}$$
$$f_e = \sum_{p \in P : e \in p} f_p, \forall e \in E$$
$$f_p \geq 0, \forall p \in P$$

By Karush-Kuhn-Tucker optimality conditions:

A feasible flow f is optimal for CP $\Leftrightarrow$ $c_p(f) \leq c_q(f)$

$$\Updownarrow$$

$$h'_p := \sum_{e \in p} (\int_0^{f_e} c_e(x)dx)' \leq \sum_{e \in q} (\int_0^{f_e} c_e(x)dx)' = h'_q,$$
$$\forall i \in \{1 \ldots k\}, \forall p, q \in P_i, f_p > 0$$

Optimal Flow

A feasible flow *f** is optimal if for every feasible flow *x*:

$$C(f^*) \le C(x) \qquad \left(C(f) = \sum_{e \in E} f_e c_e(f_e)\right)$$

Once again:

$$\begin{aligned}
&\min \sum_{e \in E} c_e(f_e) f_e \\
&\text{so that} \\
&\sum_{p \in P_i} f_p = r_i, \forall i \in \{1 \ldots k\} \\
&f_e = \sum_{p \in P : e \in p} f_p, \forall e \in E \\
&f_p \ge 0, \forall p \in P
\end{aligned}$$

By KKT conditions

$$f^* \text{ optimal} \Leftrightarrow c_p(f^*) + \sum_{e \in p} c'_e(f^*_e) f^*_e \le c_q(f^*) + \sum_{e \in q} c'_e(f^*_e) f^*_e,$$

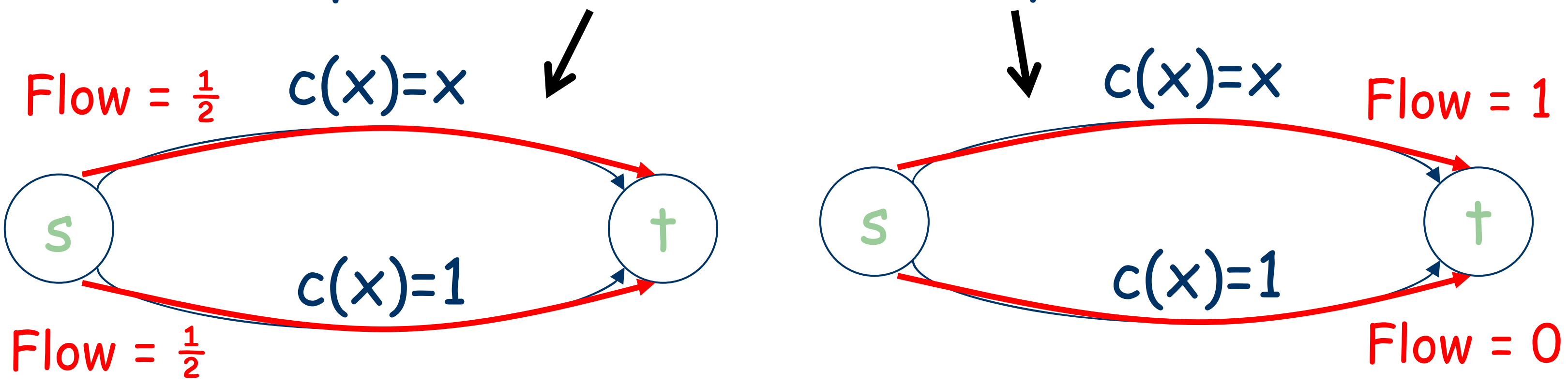
$$\forall i \in \{1 \ldots k\}, \forall p, q \in P_i, f_p > 0$$

Price of Anarchy (PoA)

A measure for the inefficiency of the network:

$\rho(G, r, c) = PoA := \frac{C(f)}{C(f^*)}$, f an equilibrium flow and f^* an optimal flow

Example: Optimal flow (OPT) and Equilirium flow (WE)



$$C(f^*) = (\tfrac{1}{2}) \cdot (\tfrac{1}{2}) + \tfrac{1}{2} \cdot 1 = \tfrac{3}{4},\ C(f) = 1 \text{ and } PoA = \frac{C(f)}{C(f^*)} = \frac{4}{3}$$

Variational Inequality

Variational inequality:

$$f \text{ Wardrop equilibrium} \Leftrightarrow \sum_{e\in E} c_e(f_e) f_e \le \sum_{e\in E} c_e(f_e) f_e^*, \forall f^* \text{ feasible}$$

- The $\Leftarrow$ part: consider *f** differing from *f* in two "same commodity" paths by *δ>0* units (for all commodities).

$$\sum_{e\in E} c_e(f_e) f_e \le \sum_{e\in E} c_e(f_e) f_e^* \Rightarrow \sum_{e\in p} c_e(f_e)\Big(f_e-(f_e-\delta)\Big) \le \sum_{e\in q} c_e(f_e)\Big((f_e+\delta)-f_e\Big)$$

- The $\Rightarrow$ part: same commodity "nonzero" paths are the cheapest of the commodity *i* and cost equal (say $c_i(f)$). Thus

$$\sum_i \sum_{p\in P_i} c_p(f) f_p = \sum_i c_i(f) \sum_{p\in P_i} f_p = \sum_i c_i(f) \sum_{p\in P_i} f_p^* = \sum_i \sum_{p\in P_i} c_i(f) f_p^* \le \sum_{p\in P} c_p(f) f_p^*$$

$$\sum_{p\in P} c_p(f) f_p \le \sum_{p\in P} c_p(f) f_p^* \Rightarrow \sum_{e\in E} c_e(f_e) f_e \le \sum_{e\in E} c_e(f_e) f_e^*$$

Bounding the PoA

Let *f* be an equilibrium flow and *f** an optimal:

$$C(f) = \sum_{e\in E} c_e(f_e)f_e \le \sum_{e\in E} c_e(f_e)f_e^* = \sum_{e\in E}\Big(c_e(f_e)f_e^* + c_e(f_e^*)f_e^* - c_e(f_e^*)f_e^*\Big) \Rightarrow$$

$$C(f) \le \sum_{e\in E} c_e(f_e^*)f_e^* + \sum_{e\in E}\big(c_e(f_e) - c_e(f_e^*)\big)f_e^* = C(f^*) + \sum_{e\in E}\big(c_e(f_e) - c_e(f_e^*)\big)f_e^*$$

We bound the last term:

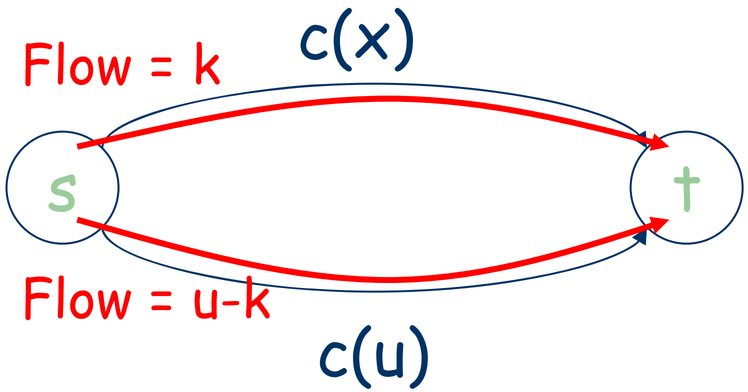
$$f_e^*\Big(c_e(f_e) - c_e(f_e^*)\Big) \le v(f_e, c_e)f_e c_e(f_e), \quad v(u, c_e) = \frac{1}{uc_e(u)} max_{x\ge 0}\{x(c_e(u) - c_e(x))\}$$

Let $v(c_e) = \sup_{u\ge 0} v(u, c_e)$ and $v(D) = \sup_{c_e} v(c_e)$ where D is the family of the cost functions. We get

$$\sum_{e\in E}\big(c_e(f_e) - c_e(f_e^*)\big)f_e^* \le v(D)\sum_{e\in E} c_e(f_e)f_e \Rightarrow C(f) \le \frac{1}{1 - v(D)} C(f^*)$$

Tightness

Assume that *u* units are to be routed from *s* to *t*.



At WE everybody goes up

OPT minimizes: $kc(k) + (u-k)c(u)$

$$PoA = \frac{uc(u)}{\min_{k\in[0,v]}[(u-k)c(u) + kc(k)]} = \max_{k\in[0,v]} \Big((1-k)+k\frac{c(k)}{uc(u)}\Big)^{-1} = \Big[1- \max_{k\in[0,v]} k\big(\frac{c(u)-c(k)}{uc(u)}\big)\Big]^{-1}$$

Previous slide: $$PoA \leq \Big(1 - \sup_{c_e\in D, u\geq 0} \max_{x\geq 0} \frac{\{x(c_e(u) - c_e(x))\}}{uc_e(u)}\Big)^{-1}$$

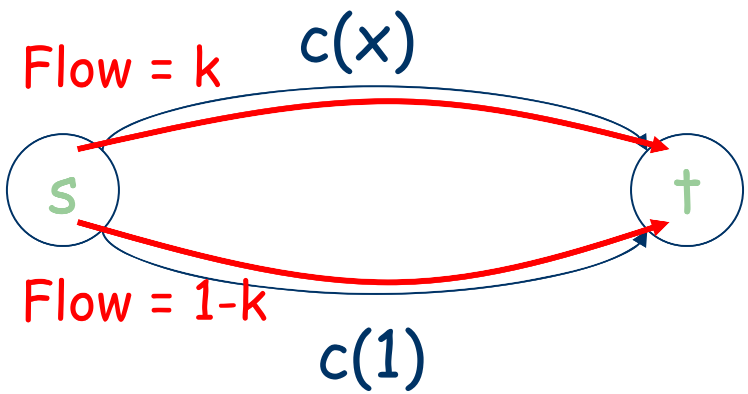
Special cases

- For linear latency functions: $v(D) = \frac{1}{4}$ and $PoA \leq \frac{4}{3}$
- For polynomial of degree *d* latency functions:

$$v(D) = \frac{d}{(d+1)^{(d+1)/d}} \text{ and } PoA \leq \left(1 - \frac{d}{(d+1)^{(d+1)/d}}\right)^{-1}$$

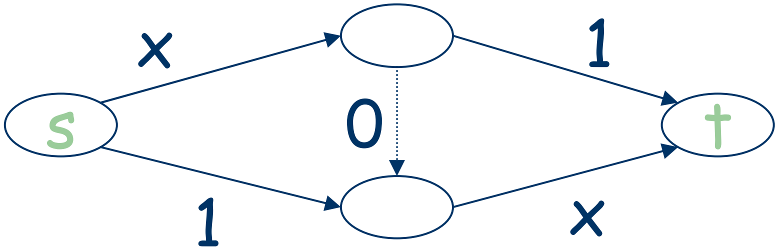
1 unit is to be routed.
At WE everybody goes up
For $c(x) = x^d$ OPT minimizes:

$$k \cdot k^d + (1-k)$$



It is $k = \sqrt[d]{\frac{1}{d+1}}$ and $OPT = 1 - \frac{d}{(d+1)^{d+1/d}}$

Braess’ Paradox



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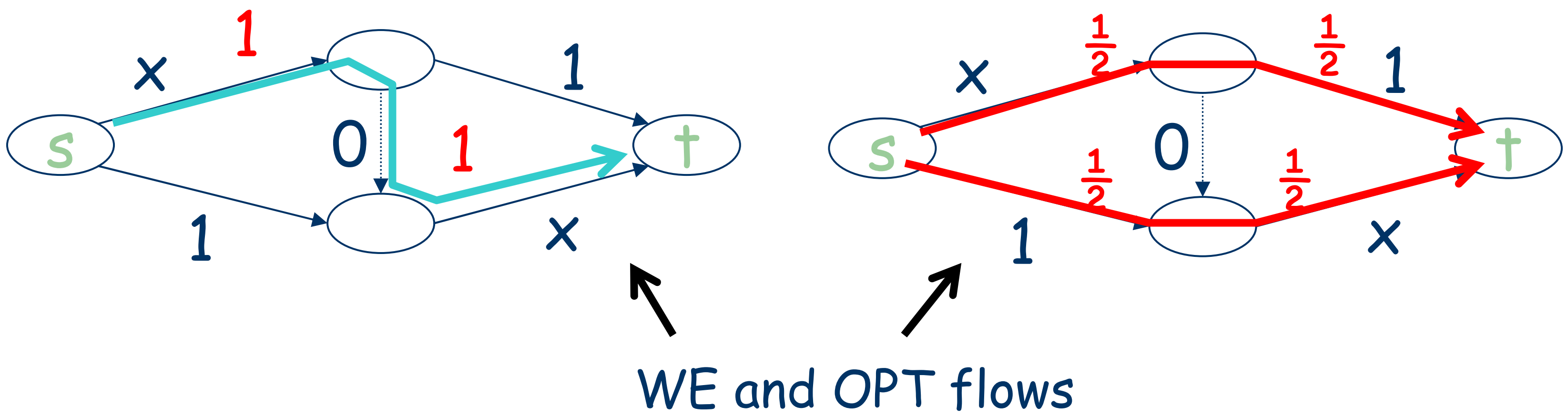

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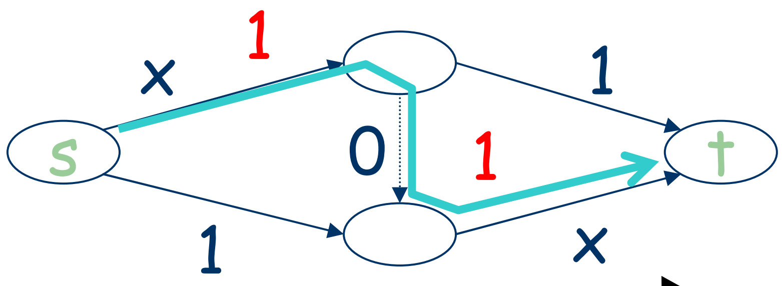
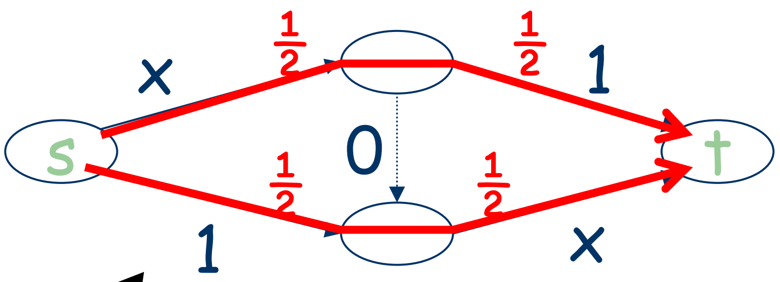
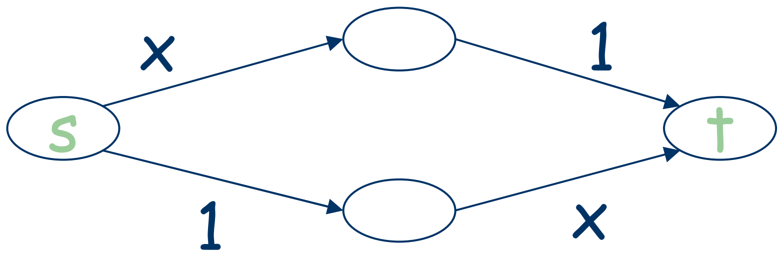




WE and OPT flows

Removing the “middle” edge:



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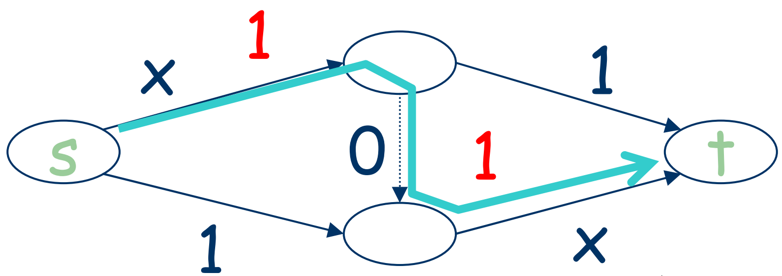
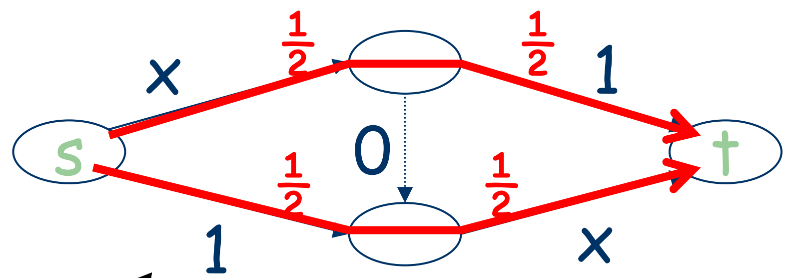
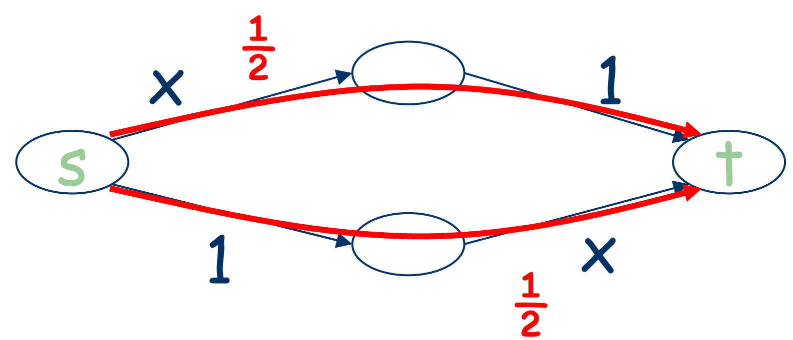




WE and OPT in the original network

Removing the "middle" edge:

OPT and WE in the subnetwork



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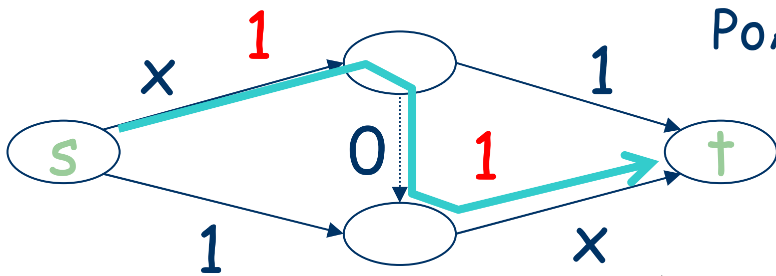
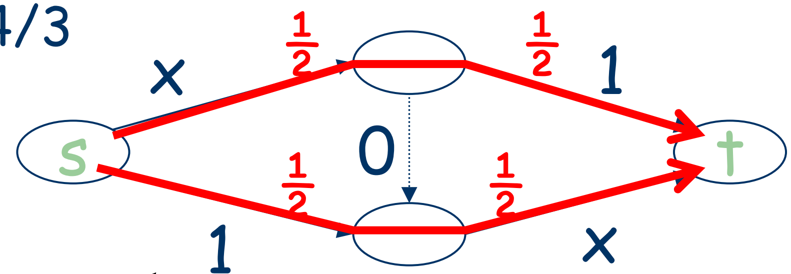


PoA=4/3



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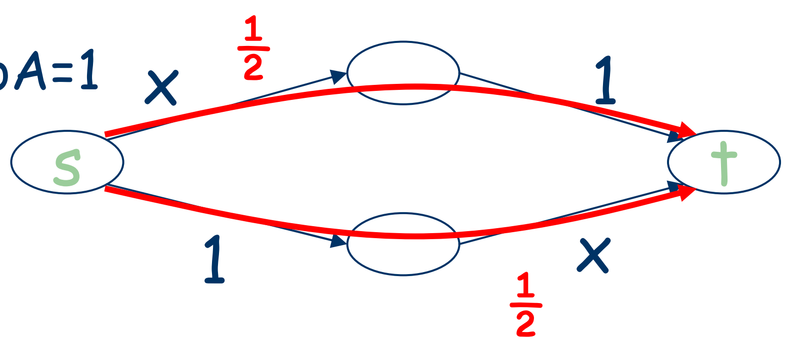
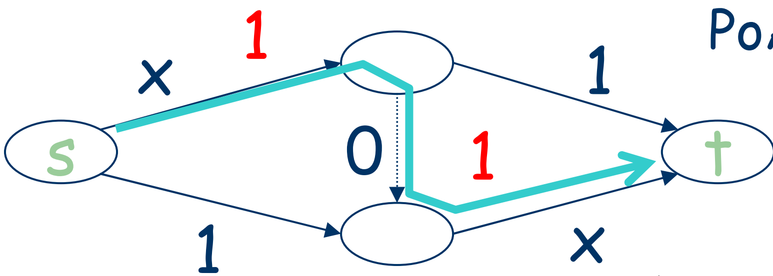
Removing the “middle” edge:

OPT and WE in the subnetwork

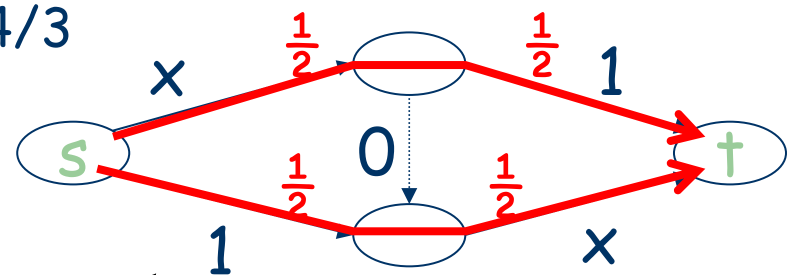
PoA=1



Braess' Paradox



PoA=4/3

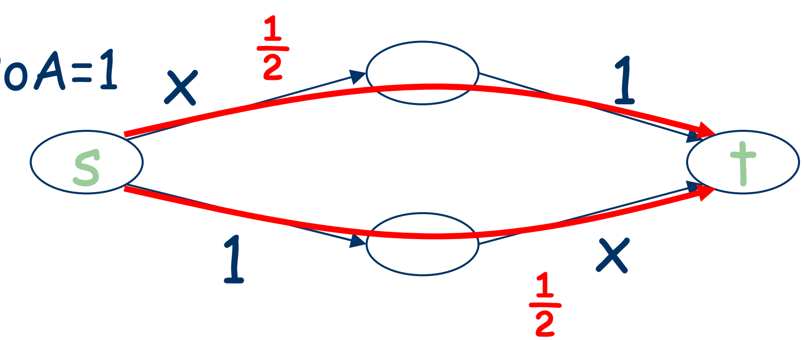


WE and OPT in the original network

Removing the "middle" edge:

OPT and WE in the subnetwork

PoA=1



Is it possible to detect the Braess' Paradox?

Reducing the PoA

The PoA can (could) be reduced:

- by detecting and excluding the **Braess' Paradox** (next time)
- by controlling a fraction of cooperative players (**Stackelberg strategies**, next time)
- by **Taxing the edges** of the network (today)
- with **Coordination Mechanisms** (or changing the rules of the game)

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Tolls

Our scope is to set tolls that “transform” the system optimum to an equilibrium

Tolls are set on the edges: edge *e* gets a τ_e

Player using path *p* gets a delay cost $c_p(f) = \sum_{e \in p} c_e(f_e)$ and has to pay $\tau_p = \sum_{e \in p} \tau_e$ as tolls.

Player *i* has a sensitivity α_i to latency. Her total cost is $a_i l_p(f) + \tau_p$

Social Cost is not affected

A "magic" LP program

Assume *g* is a (feasible) congestion that we want to enforce. Consider the following LP and its Dual:

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_i a_i \sum_{p \in P_i} c_p(g) f_p^i \\ \text{so that} \quad & \\ \forall e \in E: \quad & \textstyle\sum_i \sum_{p \in P: e \in p} f_p^i \le g_e && (1) \\ \forall i: \quad & \textstyle\sum_{p \in P_i} f_p^i = d_i && (2) \\ \forall i \forall p \in P_i: \quad & f_p^i \ge 0 && (3) \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_i d_i z_i - \sum_{e \in E} g_e t_e \\ \text{so that} \quad & \\ \forall i \forall p \in P_i: \quad & z_i - \textstyle\sum_{e \in p} t_e \le a_i c_p(g) && \text{(i)} \\ \forall e \in E: \quad & t_e \ge 0 && \text{(ii)} \end{aligned}$$

(feasible) *g* is **minimal** if inequality **1 is tight**

g **is enforceable** if there are **tolls to enforce it** on equilibrium.

The Theorem

Theorem: $\text{g minimal} \Leftrightarrow \text{g enforceable}$

Proof:

- $\Rightarrow$: there is a an optimal solution ***f***, with a complementary optimal solution ***(t,z)***, for which 1 is tight : $f_e^i > 0 \Rightarrow z_i = a_i c_p(g) + \sum_{e \in p} t_e$

$$
\begin{array}{rll}
\text{minimize} & \sum_i a_i \sum_{p \in P_i} c_p(g) f_p^i & \\
\text{so that} & & \\
\forall e \in E: & \sum_i \sum_{p \in P: e \in p} f_p^i \leq g_e & (1) \\
\forall i: & \sum_{p \in P_i} f_p^i = d_i & (2) \\
\forall i \forall p \in P_i: & f_p^i \geq 0 & (3)
\end{array}
$$

$$
\begin{array}{rll}
\text{maximize} & \sum_i d_i z_i - \sum_{e \in E} g_e t_e & \\
\text{so that} & & \\
\forall i \forall p \in P_i: & z_i - \sum_{e \in p} t_e \leq a_i c_p(g) & \text{(i)} \\
\forall e \in E: & t_e \geq 0 & \text{(ii)}
\end{array}
$$

The Theorem

Theorem: $\text{g minimal} \Leftrightarrow \text{g enforceable}$

Proof:

- $\Leftarrow$: consider eq. flow *f* and tolls τ_ε. *f* is an equilibrium:

$$f_e^i > 0 \Rightarrow z_i := a_i c_p(g) + \sum_{e \in p} \tau_e \equiv const$$

f and *(τ,z)* are **complementary** (and feasible) and so they are both **optimal**.

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_i a_i \sum_{p \in P_i} c_p(g) f_p^i \\ \text{so that} \quad & \\ \forall e \in E: \quad & \textstyle\sum_i \sum_{p \in P: e \in p} f_p^i \leq g_e \qquad (1) \\ \forall i: \quad & \textstyle\sum_{p \in P_i} f_p^i = d_i \qquad (2) \\ \forall i \forall p \in P_i: \quad & f_p^i \geq 0 \qquad (3) \end{aligned}$$

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A minimal <u>and</u> optimal g? Where?

g is called minimally feasible if:

- it is feasible and
- reducing any g_e (for any *e*) results to infeasibility

<u>A minimally feasible g has optimal solutions for which 1 is tight</u>

Let g be the optimal congestion, the one that we want to enforce.

<u>Reduce the g_e' s, stopping whenever feasibility "stops"</u>

g^* is minimally feasible + optimal ($\sum_e c_e(g^*)g^*_e \leq \sum_e c_e(g)g_e$)

Thank you!

(and Roughgarden)